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TESTING FOR COVARIANCE STATIONARITY IN STOCK MARKET DATA *

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This paper proposes several non-parametric tests for covariance stationarity and applies them to common stock return data from 1834–1987. Recursive variance plots, post-sample prediction tests, Cumulative Sum (henceforth, CUSUM) tests and modified scaled range tests all show strong non-stationarity in stock returns, primarily due to the large increase in volatility during the Great Depression. These tests should be useful as diagnostics for data where the assumptions underlying the desired statistical procedure require stationarity.

1. Introduction

Many estimators and models used in econometrics make the assumption that data are covariance stationary; that is, the mean, variance and autocovariances of a series exist and are constant over time. An example of a model requiring constancy of unconditional moments would be Hamilton's (1989) regime-shift model. Also, the existence of moments is central to the proofs of asymptotic properties of many estimators, in particular those emphasizing non-parametric methods. Neither of these assumptions can be credibly maintained without some examination of the data. Mandelbrot (1963) argued that the variance of stock returns did not exist and he suggested that one should examine a plot of the unconditional variance of returns computed recursively. If the estimate converges to a constant with increasing sample size, covariance stationarity seems a reasonable assumption. If it wanders around, it would be symptomatic of a process without a variance.

Formally, for a series y_t with zero mean, Mandelbrot's idea is to compute

$$\hat{\mu}_{2,t} = t^{-1} \sum_{j=1}^{t} y_j^2 \tag{1}$$

and to plot $\hat{\mu}_{2,t}$ against t. Although appealing, this suggestion has not been widely adopted. Indirectly there are some applications, since the widely used CUSUMs of squares test for structural change due to Brown, Durbin and Evans (1975) would be equivalent to checking the constancy of

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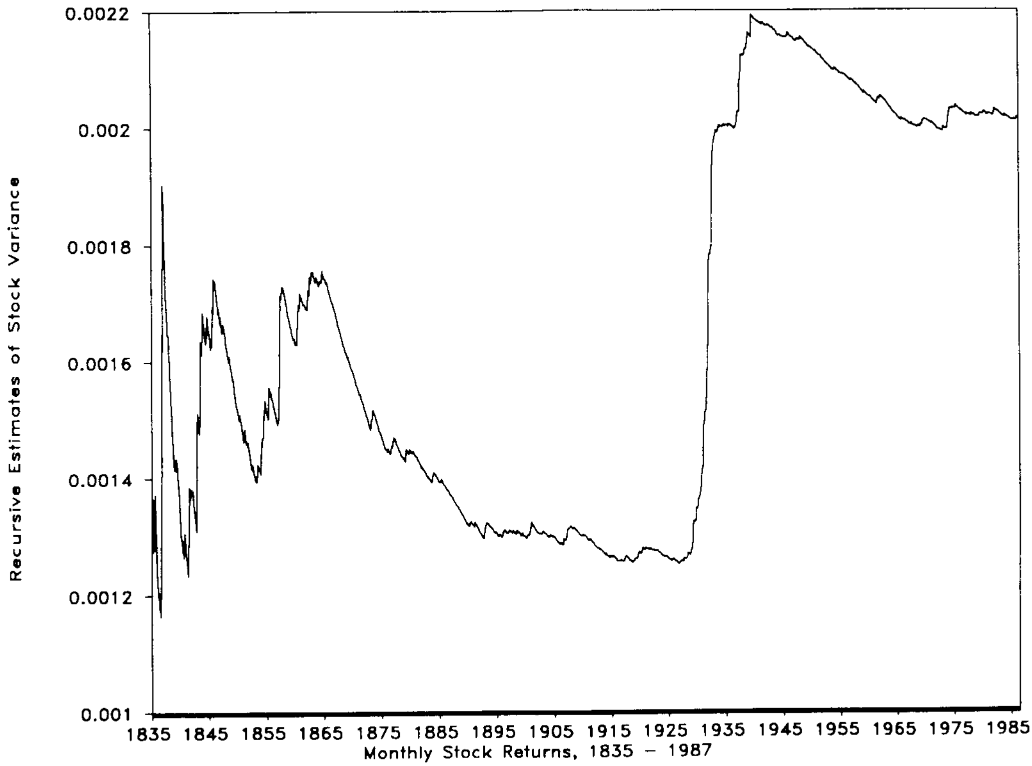

Fig. 1. Recursive estimates of the monthly stock return variance, 1835–1987.

$\hat{\mu}_{2,t}$. Some direct applications have been made by Pagan and Schwert (1989) to U.S. stock return data over the period 1835–1987 and Hols and De Vries (1989) to foreign exchange rate data. Figure 1 shows Pagan and Schwert's results, and there is dramatic evidence of a failure of covariance stationarity around the time of the Great Depression.

In our experience Mandelbrot's idea is a very useful one, but it has the disadvantage that there is no formal test statistic associated with it, so it is hard to judge either the constancy or existence of moments. As mentioned above, the closest one gets to a test is that of Brown, Durbin and Evans, but the maintained hypotheses there is that $y_t \sim N(0, \sigma^2)$, which would be inappropriate for stock market data. A lack of both normality and independence in the second moment is widely known to characterize financial asset prices, and this makes it desirable to device tests that are robust to such features. For this reason we set out a range of test statistics, based on the idea of comparing $\hat{\mu}_{2,t}$ across time. We use a variety of tests because there is no unique way to summarize the evidence. We illustrate each of these tests with the stock return data whose $\hat{\mu}_{2,t}$ is given in fig. 1. Briefly this is a monthly series on stock returns from 1834–1987 previously analyzed by Schwert (1989), who gives details of the construction of the data and places it in an historical context. Our only transformation to the data is to eliminate calendar effects by regressing out twelve monthly dummies, so y_t are actually the residuals from such a regression.

2. Tests for covariance stationarity

As mentioned in the introduction, there is no unique way to construct tests for homogeneity of variances. In this section we therefore propose three different approaches.

2.1. Post-sample prediction tests for covariance stationarity

One possibility is to split the sample into two parts and to compare the sample variance $\hat{\mu}_2^{(1)}$ and $\hat{\mu}_2^{(2)}$ of each sample. Let the sample be split such that $T = T_1 + T_2$, $T_2 = kT_1$, and consider testing the hypothesis that

$$\mathrm{E}\left[T_1^{-1}\sum_{j=1}^{T_1} y_j^2\right] = \mathrm{E}\left[T_2^{-1}\sum_{j=T_1+1}^{T} y_j^2\right]. \tag{2}$$

A suitable test statistic is $\hat{\tau} = \hat{\mu}_2^{(2)} - \hat{\mu}_2^{(1)}$ and one can think of this as a member of the class of 'post-sample prediction tests' studied by Ghysels and Hall (1989) and Hoffman and Pagan (1989). Setting $k = 1$, it follows from those papers that

$$T_1^{1/2}\hat{\tau} \xrightarrow{\mathrm{d}} \mathrm{N}\left(0,\ 2\left(\gamma_0 + 2\sum_{j=1}^{\infty}\gamma_j\right)\right) \tag{3}$$

if y_t^2 is a covariance stationary process with autocovariances γ_j that obey certain mixing conditions. [1] The essential fact for establishing the limiting distribution is that

$$\left(T_1^{-1/2}\sum_{j=1}^{T_1} y_j^2\right) \quad \text{and} \quad \left(T_2^{-1/2}\sum_{j=T_1+1}^{T} y_j^2\right), \tag{4}$$

are asymptotically uncorrelated, while $\hat{\mu}_2^{(1)}$ and $\hat{\mu}_2^{(2)}$ have the same probability limits when y_t^2 is covariance stationary. Since

$$\nu = \left(\gamma_0 + 2\sum_{k=1}^{\infty}\gamma_k\right) \tag{5}$$

is proportional to the spectral density of y_t^2 at the origin one could consistently estimate it this way. Instead, we follow Phillips (1987) and estimate it by

$$\hat{\gamma}_0 + 2\sum_{j=1}^{8}\hat{\gamma}_j(1 - (j/9)), \tag{6}$$

where $\hat{\gamma}_j$ are the estimated serial correlation coefficients of y_t^2 calculated over the whole sample. For the complete sample of 1834–1987, $\hat{\mu}_2^{(1)} = 0.0013$, $\hat{\mu}_2^{(2)} = 0.0028$ and the 't-statistic'

$$T_1^{1/2}\left(\hat{\tau}/\sqrt{2\hat{\nu}}\right) = -3.00, \tag{7}$$

is showing a lack of homogeneity in the variance.

[1] Our series for y_t is really the residuals from a regression. It is easily shown that this does not affect the limiting distribution of any of the tests discussed here.

2.2. CUSUM test for covariance stationarity

Something close to the recursive calculation in fig. 1 is to examine the cumulative sums of $(y_t^2 - \tilde{\mu}_2)$, where $\tilde{\mu}_2$ is the variance estimated over the whole period. Thus, define

$$\psi(r) = (T\hat{\nu})^{-1/2} \sum_{j=1}^{[Tr]} \left(y_j^2 - \tilde{\mu}_2\right), \tag{8}$$

where $0 < r < 1$, $[\cdot]$ is the 'integer part of' [using the notation in Phillips (1987)] and $\tilde{\mu}_2 = T^{-1}\Sigma_{t=1}^{T} y_t^2$. We therefore need to find the limiting distribution of $\psi(r)$ if we are to find $\mathrm{pr}(\psi(r) > c)$. Denoting $\mu_2 = \mathrm{E}(y_j^2)$, $\psi(r)$ can be decomposed as

$$\psi(r) = (T\nu)^{-1/2}\left\{\sum_{j=1}^{[Tr]}\left(y_j^2 - \mu_2 + \mu_2 - \tilde{\mu}_2\right)\right\}$$

$$= (T\nu)^{-1/2}\left\{\sum_{j=1}^{[Tr]}\left(y_j^2 - \mu_2\right) + [Tr]\left(T^{-1}\sum_{j=1}^{T}\left(\mu_2 - y_j^2\right)\right)\right\}$$

$$= (T\nu)^{-1/2}\left\{\sum_{j=1}^{[Tr]}\phi_j - r\sum_{k=1}^{[Tr]}\phi_k - r\sum_{m=[Tr]+1}^{T}\phi_m\right\} \quad \text{where} \quad \phi_j = y_j^2 - \mu_2,$$

$$= (1-r)(T\nu)^{-1/2}\sum_{j=1}^{[Tr]}\phi_j - r(T\nu)^{-1/2}\sum_{k=[Tr]+1}^{T}\phi_k. \tag{9}$$

If ϕ_j were normally and independently distributed, then the two terms in (9) would be independent. The first of these terms would be an $(1-r)\mathrm{N}(0, r)$ random variable and the second an $r\mathrm{N}(0, 1-r)$ random variable. This outcome can be generalized to the case when the y_j are non-normal and obey the moment and mixing conditions in Phillips (1987). Lo (1987) formally proves that $\psi(r)$ converges in distribution to a Brownian bridge under these conditions, making $\Pr(\psi(r) < c)$ equal to the probability that an $\mathrm{N}(0, r(1-r))$ random variable is less than c. [2]

The testing strategy is similar to that in Brown, Durbin and Evans (1975), except that recursive residuals are not used and we have centered the test statistic to use an invariance principle. Because this test might be useful in a wider context, table 1 contains several fractiles $c^+(r)$ and $c^-(r)$ of the Brownian bridge for $r = 0.1, 0.2, 0.3, \ldots, 0.9$. Figure 2 shows $\psi(t)$ plotted against t for 1834–1987 along with the lower limit for the 99 percent confidence interval $c_{0.01}^-(r)$ to indicate how $c_{0.01}(r)$ varies with r. There is little doubt about a lack of covariance stationarity over the complete sample. The minimum value of $\hat{\psi}(t)$ is -2.51, which is much smaller than $c_{0.01}^-(r)$.

2.3. Modified scaled range test for covariance stationarity

Another possible test statistic for homogeneity is to compare max $\psi(r)$ with min $\psi(r)$, $0 < r < 1$. The statistic

$$R = \max_{0<r<1} \psi(r) - \min_{0<r<1} \psi(r) \tag{10}$$

[2] We thank Andrew Lo for pointing out that the distribution of the Brownian bridge is normal.

Table 1
Fractiles of the Brownian bridge for various fractions r of the total sample. [a]

Fraction of r	Fraction of T					
	0.005	0.025	0.05	0.95	0.975	0.995
0.1	−0.773	−0.588	−0.493	0.493	0.588	0.773
0.2	−1.030	−0.784	−0.658	0.658	0.784	1.030
0.3	−1.180	−0.898	−0.754	0.754	0.898	1.180
0.4	−1.262	−0.960	−0.806	0.806	0.960	1.262
0.5	−1.288	−0.980	−0.822	0.822	0.980	1.288
0.6	−1.262	−0.960	−0.806	0.806	0.960	1.262
0.7	−1.180	−0.898	−0.754	0.754	0.898	1.180
0.8	−1.030	−0.784	−0.658	0.658	0.784	1.030
0.9	−0.773	−0.588	−0.493	0.493	0.588	0.773

[a] These fractiles are based on the cumulative normal density with a variance $r(1-r)$. The 0.005 and 0.995 fractiles are the lower and upper limits for a 1 percent level test, $c^{-}_{0.01}(r)$ and $c^{+}_{0.01}(r)$.

is termed the modified scaled range statistic by Haubrich and Lo (1988), in recognition of its origin in the scaled range statistic of Mandelbrot (1972). For 1834–1987, $R = 2.36$, and from Haubrich and Lo (1988, table 1a), $\Pr[R \geq 2.36] < 0.005$, reinforcing our previous conclusion that there is strong evidence of a lack of covariance stationarity over the complete sample.

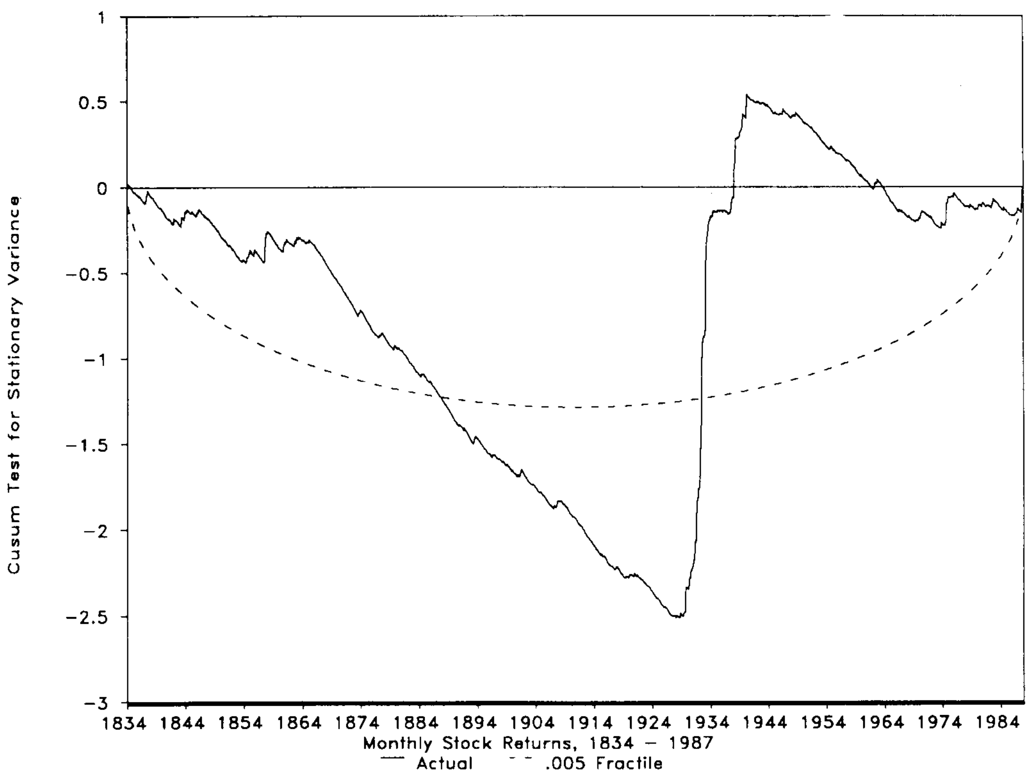

Fig. 2. CUSUM test for stationarity of stock return variance, 1834–1987 (with 0.005 fractile from the sampling distribution).

3. Conclusion

Our paper has aimed at presenting some ways for assessing whether a series exhibits covariance stationarity by exploiting information based on recursive estimates of the conditional variance. In all our proposed tests, monthly stock returns exhibited a failure of covariance stationarity over the period 1834–1987, calling into question any models and estimators employing such data and which rely on covariance stationarity. For example, it would be meaningless to compute an autocorrelation function. Because the tests rejected at high levels of significance we were not concerned about their power, and simulations (not reported) showed that correspondence of asymptotic and nominal significance levels was good with this large data set. However, if our tests are to be applied more generally, it would be useful to assess their power properties both analytically and by means of Monte Carlo experiments.

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